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DEPARTMENT OF ELECTRICAL ENGINEERING

ACOUSTICS LABORATORY

ON THE PROPAGATION OF SOUND
IN A
TURBULENT FLUID

By

ROBERT H. KRAICHNAN

Submitted by

Cyril M. Harris

Project Director

CU4-54 — ONR266(23)-EE

March 11, 1954

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Abstract

This report consists of three parts. In Part 1, the eikonal and continuity equations are derived for a sound wave propagating through a fluid in which there is shear motion of low Mach number and scale large compared to the sound wave length. On the basis of a comparison between the eikonal equation and the Hamilton-Jacobi equation for a charged particle in a magnetic field it is shown that the acoustic ray paths are identical with the trajectories of the particles if the magnetic field is proportional to the vorticity.

In Part 2, the attenuation and fluctuation in intensity of a sound beam scattered in a turbulent medium are expressed in terms of correlation products of the turbulent flow. Expressions are also derived for the phase and intensity fluctuations associated with the propagation of sound through large scale turbulence, as discussed in Part 1.

In Part 3, the differential cross section for scattering of sound by turbulence is developed in a form suited to treatment of anisotropic turbulence. A simple anisotropic distribution involving symmetry about a preferred vorticity axis is discussed.

Contents

Notation

Both vector and tensor notations are employed
according to convenience. Vectors are indicated by
a wavy underline. Repeated tensor indices are to be
summed. Sub- or superscripts in parentheses are not
tensor indices. The anticommuting symbol $\epsilon_{\alpha\beta\tau}$ $(\alpha,\beta,\tau=1,2,3)$
has the value one for $\alpha=1,\beta=2,\tau=3$, is antisymmetric
to permutation of any pair of indices, and vanishes if
two indices are equal. In terms of this symbol the
cross product of the vectors A_μ and B_λ is $C_\sigma = \epsilon_{\sigma\mu\lambda}A_\mu B_\lambda$.

Symbols of the form d^3x indicate integration over
three dimensional space. The symbol $d^2\underset{\sim}{n}$ indicates in-
tegration over all angles.

Space derivatives such as $\partial f/\partial x^\alpha$, $\partial^2 f/\partial x^\alpha \partial x^\beta$ are
frequently denoted by $f,_\alpha$ and $f,_{\alpha\beta}$ respectively.

Part 1.   Propagation of Sound through Large Scale Turbulence

1.1   When a sound wave propagates through a fluid in shear
motion, energy is scattered out of the beam[1] if the spatial
power spectrum of the shear motion, or turbulence, has appreci-
able strength at wave numbers which are the same order of size
as that of the sound wave. When the dominant eddy size in the
shear flow is considerably larger than the sound wave length such
scattering becomes very small, but there remains a coherent type
of perturbation which results in gradual distortion of the wave
fronts without loss of energy. This distortion can be described
by a WKB type approximation procedure which yields an eikonal
equation determining the phase fronts and a continuity equation
describing the amplitude fluctuations associated with the phase
distortion.

In the presence of a shear velocity field of low Mach
number the wave equation for sound propagation has the form[2]

$$\Box^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = \frac{2 P_0}{c^2}\frac{\partial^2 (\mathcal{U}_\alpha W_\beta)}{\partial x^\alpha \partial x^\beta} \tag{1.1}$$

where p is the excess pressure, P_0 the mean pressure, $\underset{\sim}{\mathcal{U}}$ the
longitudinal or sound velocity obeying

$$\mathcal{U}_{\alpha,\beta} - \mathcal{U}_{\beta,\alpha} = 0 \tag{1.2}$$

and $\underset{\sim}{W}$ the shear or transverse velocity obeying

$$W_{\alpha,\alpha} = 0 \tag{1.3}$$

The velocity of sound in the undisturbed medium is c.

(1)   Kraichnan, Robert H., J.A.S.A.$\underline{25}$ 1096 (1953)

(2)   Lighthill, M. J., Proc. Roy. Soc. A211 564 (1952),
        Proc. Camb. Phil. Soc. 49 531 (1953)

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In the absence of the shear velocity $\underset{\sim}{w}$, the pressure and velocity for a weak plane sound wave of arbitrary form may be taken as

$$p = p_0 A f(t - n_\alpha x^\alpha) \qquad u_\alpha = c A n_\alpha f(t - n_\alpha x^\alpha) \qquad (1.4)$$

where A is a dimensionless constant which we shall call the amplitude. We now try to satisfy (1.1), under the assumption that the Mach number of the shear flow is small and that $\underset{\sim}{w}$ varies but little in a sound wavelength, by replacing A with the space function $A(\underset{\sim}{x})$ and $n_\alpha x^\alpha$ with the function $\vartheta(\underset{\sim}{x})$. The usefulness of this approach is that because $\underset{\sim}{w}$ varies slowly, the variations in A and ϑ from their unperturbed forms will also be slow.

By making use of the divergence and curl conditions (1.3) and (1.2), the right side of (1.1) may be written

$$\frac{2p_0}{c^2} \frac{\partial^2 (u_\alpha w_\beta)}{\partial x^\alpha \partial x^\beta} = \frac{2p_0}{c^2} (w_\alpha \nabla^2 u_\alpha + u_{\alpha,\beta} w_{\alpha,\beta})$$

Since, because of the low Mach number associated with $\underset{\sim}{w}$, this term is a small perturbation, we may replace $\nabla^2 u_\alpha$ by the unperturbed value $c^{-2} \partial^2 u_\alpha / (\partial t)^2$ within the limits of the approximations to which (1.1) is valid.

We shall now substitute into (1.1) the functions

$$p = p_0 A(\underset{\sim}{x}) f(t - \vartheta(\underset{\sim}{x})) \qquad u_\alpha = c A(\underset{\sim}{x}) \vartheta_{,\alpha} f(t - \vartheta(\underset{\sim}{x})) \qquad (1.5)$$

where $\vartheta_{,\alpha}$ stands for $\partial \vartheta / \partial x^\alpha$. The result is

$$A f'' - \vartheta_{,\alpha} \vartheta_{,\alpha} A f'' + \nabla^2 \vartheta f' A + 2 A_{,\alpha} \vartheta_{,\alpha} f' - \nabla^2 A f \qquad (1.6)$$

$$= 2 c^{-1} [w_\alpha \vartheta_{,\alpha} A f'' + w_{\alpha,\beta} (\vartheta_{,\alpha\beta} A f - \vartheta_{,\alpha} \vartheta_{,\beta} A f' + \vartheta_{,\alpha} A_{,\beta} f)]$$

where the prime denotes differentiation with respect to the argument. The quantities $A_{,\alpha}$, $\vartheta_{,\alpha\beta}$, w_α, $w_{\alpha,\beta}$ are small quantities in this equation according to the assumption of slow variation of the shear velocity. Hence, neglecting terms bilinear in these quantities we obtain the approximate equation

$$A f'' - \vartheta_{,\alpha} \vartheta_{,\alpha} A f'' + \nabla^2 \vartheta f' A + 2 A_{,\alpha} \vartheta_{,\alpha} f' = 2 c^{-1} [w_\alpha \vartheta_{,\alpha} A f'' - w_{\alpha,\beta} \vartheta_{,\alpha} \vartheta_{,\beta} A f'] \qquad (1.7)$$

We have also neglected the term in $\nabla^2 A$, which will be small compared to $A,_\alpha$. Since f is a quite arbitrary function, the coefficients of f'' and f' must be independently zero; we therefore obtain the two equations

$$\vartheta,_\alpha \vartheta,_\alpha + 2c^{-1}\vartheta,_\alpha W_\alpha - 1 = 0 \tag{1.8}$$

$$2A,_\alpha \vartheta,_\alpha + A\nabla^2\vartheta = -2c^{-1}\vartheta,_\alpha \vartheta,_\beta W_{\alpha,\beta} A \tag{1.9}$$

The first of these relations is an eikonal equation determining the phase as a function of space. Rewriting the second in the form

$$(\log A^2),_\alpha \vartheta,_\alpha + \nabla^2\vartheta + 2c^{-1}\vartheta,_\alpha \vartheta,_\beta W_{\alpha,\beta} = 0 \tag{1.10}$$

it is seen to be a continuity equation relating square of amplitude to phase. Eq. (1.10) shows that the variations in amplitude arise from two causes: focusing effects associated with the curvature of the wavefronts; and the necessity for maintenance of constant energy flow through regions in which the local velocity of propagation varies in the direction of propagation.

1.2   In this section it will be shown that the effect on the propagation of sound rays of large eddy size turbulence, or, in general, shear motion which varies but little in the space at a sound wave length, bears a direct analogy to the effect of magnetic fields on the paths of charged particles. Since the latter has been extensively studied in connection with fields such as cosmic ray research and particle accelerator design, the analogy possibly could be a valuable tool in visualizing sound propagation phenomena. The analogy is based on a similarity between the eikonal equation for sound waves in a slowly moving medium and the Hamilton-Jacobi equation for the motion of charged particles in a magnetic field.

      Consider a particle whose motion is determined by the Hamilton equations

$$\dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha} \qquad\qquad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha} \tag{1.11}$$

where q_α is the position vector of the particle, $\dot{q}_\alpha$ the time rate of change of this vector. The Hamiltonian H and momenta p_α are

related to a time-independent Lagrangian function $L(q_\alpha, \dot{q}_\alpha)$ by

$$ p_\alpha = \partial L / \partial \dot{q}_\alpha \qquad\qquad H(p_\alpha, q_\alpha) = \sum_\alpha p_\alpha \dot{q}_\alpha - L \qquad (1.12) $$

The possible motions of such a particle corresponding to a total energy E may then be found, according to the Hamilton-Jacobi procedure[3], by solving for S the equation

$$ H\left(\frac{\partial S}{\partial q_\alpha}, q_\alpha\right) = E \qquad (1.13) $$

in which $\partial S/\partial q_\alpha$ replaces p_α wherever it appears in the Hamiltonian.

The paths along which the particles of energy E travel may be described very simply in terms of the function S. According to Hamilton's principle of least action[4] the particle paths determined by (1.13) are those for which the action integral

$$ A = \int_{t^{(1)}}^{t^{(2)}} \sum_\alpha p_\alpha \dot{q}_\alpha \, dt \qquad (1.14) $$

is extremal (maximum or minimum) between two arbitrary fixed end points $q_\alpha^{(1)}$ and $q_\alpha^{(2)}$. Regarding the time as a parameter along the path, we may write $\dot{q}_\alpha dt = dq_\alpha$ so that, substituting $\partial S/\partial q_\alpha$ for p_α, the extremal condition on (1.14) may be expressed as

$$ \delta \int_{q_\alpha^{(1)}}^{q_\alpha^{(2)}} \sum \frac{\partial S}{\partial q_\alpha} \, dq_\alpha = 0 \qquad (1.15) $$

If now a wave pattern were constructed such that S were everywhere proportional to the total phase of the waves, (1.15) would determine those paths between the two fixed end points along which the total change of phase would be extremal. It is therefore a statement of Fermat's principle, and the paths so defined are simply the ray paths of the wave system as ordinarily understood.

We have therefore obtained the general result that the ray paths of a wave system are identical with a family of trajectories of particles obeying a Hamilton-Jacobi equation identical with the eikonal equation for the waves. This fact, which is the basis of the correspondence between classical and quantum mechanics, takes a simple and well known form in the case where the Hamiltonian contains no velocity dependent forces. Then,

(3)   See, e.g., Goldstein, "Classical Mechanics" Addison-Wesley, Cambridge (1951)

(4)   See reference (3) above

the particle velocity $\dot{q}_\alpha$ is proportional to the canonical momentum p_α so that the ray paths are normal to the phase fronts, S = constant, and the ray direction at any point is the direction of the vector $\partial S / \partial q_\alpha$.

The Lagrangian of a non-relativistic particle of mass m and charge e in an electromagnetic field with vector potential $A_\alpha(q)$ and scalar potential $V(q)$ is

$$L = \frac{m \dot{q}_\alpha \dot{q}_\alpha}{2} + \frac{e \dot{q}_\alpha A_\alpha}{c_0} - eV \qquad (1.16)$$

where c_0 is the velocity of light.

The corresponding momenta and Hamiltonian are

$$p_\alpha = m \dot{q}_\alpha + \frac{e}{c_0} A_\alpha$$

$$H(p,q) = (p_\alpha - \frac{e}{c_0} A_\alpha)^2 / 2m + eV \qquad (1.17)$$

Accordingly the Hamilton-Jacobi equation is

$$\left(\frac{\partial S}{\partial q_\alpha} - \frac{e}{c_0} A_\alpha\right)^2 / 2m + eV = E \qquad (1.18)$$

where E is the total energy of the particle.

The approximate eikonal equation for a sound wave propagating in a medium undergoing shear flow was found to be

$$\vartheta_{,\alpha} \vartheta_{,\alpha} + 2 c^{-1} \vartheta_{,\alpha} W_\alpha - 1 = 0 \qquad (1.8)$$

according to section 1.1 Here $\omega \vartheta / c$ is the total phase, ω is the frequency, c the velocity of sound in the still medium, and W_α the shear velocity field obeying

$$W_{\alpha,\alpha} = 0 \qquad (1.3)$$

Since (1.8) is accurate only to terms linear in W_α, it may be written to equally good approximation as

$$(\vartheta_{,\alpha} + c^{-1} W_\alpha)^2 = 1 \qquad (1.19)$$

which, it may be verified, differs from (1.8) by only the term $c^{-2}|\underset{\sim}{w}|^2$.

In this form, the similarity to the Hamilton-Jacobi equation (1.18) with $V=0$ is readily apparent. The similarity is made more complete by the fact that the vector potential obeys $A_{\alpha,\alpha}=0$, when V=constant, in analogy to (1.3). The exact identification between (1.18) and (1.19) may be shown by dividing (1.18) by E, and noting that $2mE$ is simply the square of the ordinary momentum of the particle, which we may denote by P^2. In terms of this quantity (1.18) may be written

$$\left(S_{,\alpha} - e c_o^{-1} A_\alpha\right)^2 / P^2 = 1 \tag{1.20}$$

which is identical with (1.19) if

$$S_{,\alpha}/P = \mathcal{Q}_{,\alpha} \qquad - e c_o^{-1} A_\alpha /P = c^{-1} w_\alpha \tag{1.21}$$

Thus, paths of sound rays in a fluid in which the local velocity is w_α are identical with the trajectories of particles of charge e and momentum P moving in an electromagnetic field corresponding to the vector potential $A_\alpha = c_o c^{-1} P w_\alpha /e$.

Several limitations on the analogy must be kept in mind. First we have assumed that the velocity field w_α changes but little in a distance of a sound wavelength, and is small compared to c. Secondly we have not as yet included the effects of time variation of w_α; our discussion of the Hamilton-Jacobi equation was based on the assumption of time-independent potentials. Actually, the requirements of large eddy size and small Mach number imply that the time-variations in w_α will be very slow indeed compared to those of the sound wave and except for special cases they will not result in significant departure from the time independent equations.

The magnetic field corresponding to the vector potential in (1.21) is

$$\underset{\sim}{H} = curl\, \underset{\sim}{A} = - \left(c_o c^{-1} P/e\right) curl\, \underset{\sim}{w}$$

or

$$e\underset{\sim}{H}/c_o P = - c^{-1}\underset{\sim}{\mathcal{V}} \tag{1.22}$$

where $\underset{\sim}{\mathcal{V}}$ is the vorticity associated with $\underset{\sim}{w}$. Hence the results of this section may be summed up by saying that, within the limitations noted above, the ray paths of a sound wave in a

moving fluid follow the trajectories of charged particles in a magnetic field everywhere proportional to and in the opposite direction from the vorticity vector, the proportionality constant being determined in terms of the (arbitrary) particle momentum by (1.22).[5]

It was noted above that every turbulent velocity field $\underset{\sim}{w}$ corresponds to a possible magnetic field in free space, and the converse is also kinetically possible. Hence, the propagation of sound in a moving atmosphere should exhibit all the focussing effects, shadow effects, and other phenomena encountered with particles travelling in magnetic fields. It would be of interest to examine to what extent the analogy holds true when eddy sizes the order of a wavelength or smaller are involved. This would correspond, if the analogy held, to the quantum mechanics of charged particles in magnetic fields. Also of interest is the question of whether such time-dependent phenomena as the betatron acceleration mechanism have counterparts in the sound propagation case. To investigate this it is necessary to extend the treatment above to include time dependent potentials. There could possibly result an analogy between the ray paths for the time dependent case and the trajectories of the 4-dimensional Hamilton-Jacobi equation for a relativistic particle.

Part 2. Attenuation and Intensity Fluctuations

2.1   When a sound wave propagates through a turbulent fluid medium, the local intensity of the wave is altered both by scattering of energy out of the beam, as discussed by the author in a previous paper,[1] and by distortion of the shape of the wavefronts by large scale shear flow as indicated in the previous section. In this section, the intensity variations to be expected from both effects will be expressed in terms of the statistical correlation products associated with the turbulence. The calculations will be confined to cases in which the characteristic time periods of the turbulence are very large compared to the period of the sound wave; this represents a realistic restriction for the discussion of sound propagation in the atmosphere or sea, where very small Mach numbers are ordinarily encountered.

In reference (1) expressions were derived for the differential scattering cross-section for sound waves traversing a region in turbulent motion. It follows from the analysis developed there that when the Mach number of the turbulence is

(5)   Note that there is no direct relation between the speed of propagation of the rays and the speed of the particles. Only the spatial forms of the paths are compared.

small, the frequency spectrum of the differential scattering cross-section is given by

$$I(\underline{n}^{(i)}, \underline{n}^{(2)}, \omega) = \frac{\pi \omega^4}{c^6}(\underline{n}^{(i)} \cdot \underline{n}^{(2)})^2 \left| \underline{n}^{(2)} \cdot \underline{W}\left(\frac{|\omega| \underline{n}^{(2)} - \underline{n}^{(i)} \omega_\bullet}{c}, |\omega| - \omega_\bullet\right)\right|^2 \quad (2.1)$$

Here c is the velocity of sound, $\underline{n}^{(i)}$ a unit vector in the direction of the incident sound wave, $\underline{n}^{(1)}$ a unit vector in a direction of scattering, $\omega_\bullet$ the frequency of the incident sound-wave, and $\underline{W}(\underline{k}, \omega)$ the 4-dimensional Fourier transform of the turbulent velocity field. $I(\underline{n}^{(i)}, \underline{n}^{(2)}, \omega)$ is related to the differential scattering cross-section by

$$\sigma(\underline{n}^{(i)}, \underline{n}^{(2)}) = \frac{1}{\tau} \int_{-\infty}^{\infty} I(\underline{n}^{(i)}, \underline{n}^{(2)}, \omega) \, d\omega \quad (2.2)$$

where τ is the period of time over which the time Fourier analysis is performed.

The principal effects on the scattering of the time variation of the turbulence as reflected in the 4-dimensional transform $\underline{W}(\underline{k}, \omega)$ are first, a slight broadening in frequency of the scattered wave, and second, an effect on the amount of scattering at very small angles in radians of the order of magnitude of the ratio of the characteristic frequencies of the turbulence to $\omega_\bullet$. These effects will have a negligible influence on the attenuation of a sound beam, which depends on the total cross-section and hence we shall express the scattering cross-section in the time independent form

$$\sigma(\underline{n}^{(i)}, \underline{n}^{(2)}) = \frac{\pi}{c^2} k_\bullet^4 (\underline{n}^{(i)} \cdot \underline{n}^{(2)})^2 \left| \underline{n}^{(2)} \cdot \underline{W}(k_\bullet[\underline{n}^{(2)} - \underline{n}^{(i)}])\right|^2 \quad (2.3)$$

where $k_\bullet = \omega_\bullet / c$ and $\underline{W}(\underline{k})$ the 3-dimensional Fourier transform of the shear velocity is defined by

$$\underline{W}(\underline{k}) = (2\pi)^{-\frac{3}{2}} \int \underline{w}(\underline{x}) e^{-i\underline{k} \cdot \underline{x}} \, d^3\underline{x} \quad (2.4)$$

Equation (2.3) can be derived by performing the analysis of reference (1) under the initial assumption that the shear velocity is independent of time. In using this expression to calculate the attenuation of a sound wave over long distances, it is assumed, of course, that the turbulence is statistically stationary over the regions and times involved.

The scattering cross-section as it appears in (2.3) may be expressed readily in terms of the correlation products usually employed in describing turbulence. Rewriting the expression in $\underline{W}(\underline{k})$,

$$\sigma(\underline{n}^{(1)},\underline{n}^{(2)}) = \frac{\pi}{c^2} k_o^4 (\underline{n}^{(1)}\cdot\underline{n}^{(2)})^2 n_\alpha^{(2)} n_\beta^{(2)} W_\alpha^*(k_o[\underline{n}^{(2)}-\underline{n}^{(1)}]) W_\beta(k_o[\underline{n}^{(2)}-\underline{n}^{(1)}]) \quad (2.5)$$

If the correlation tensor of the turbulence is denoted by

$$t_{\alpha\beta}(\underline{\lambda}) = \overline{w_\alpha(\underline{x}) w_\beta(\underline{x}+\underline{\lambda})} \quad (2.6)$$

where the overbar indicates averaging over a suitable region of space, the Fourier transform of $t_{\alpha\beta}$,

$$T_{\alpha\beta}(\underline{k}) = (2\pi)^{-\frac{3}{2}} \int t_{\alpha\beta}(\underline{\lambda}) e^{-i\underline{k}\cdot\underline{\lambda}} d^3\underline{\lambda} \quad (2.7)$$

differs from $w_\alpha^*(\underline{k})w_\beta(\underline{k})$ only in that it is normalized by the total volume of turbulence considered. Hence, the scattering cross-section per average cubic centimeter of turbulence will be

$$\bar{\sigma}(\underline{n}^{(1)},\underline{n}^{(2)}) = \frac{\pi}{c^2} k_o^4 (\underline{n}^{(1)}\cdot\underline{n}^{(2)})^2 n_\alpha^{(2)} n_\beta^{(2)} T_{\alpha\beta}(k_o[\underline{n}^{(2)}-\underline{n}^{(1)}]) \quad (2.8)$$

The total scattering cross-section is found by integrating (2.8) over all scattering directions $\underline{n}^{(2)}$:

$$\bar{\sigma}(\underline{n}^{(1)}) = \frac{\pi}{c^2} k_o^4 \int (\underline{n}^{(1)}\cdot\underline{n}^{(2)})^2 n_\alpha^{(2)} n_\beta^{(2)} T_{\alpha\beta}(k_o[\underline{n}^{(2)}-\underline{n}^{(1)}]) d^2\underline{n}^{(2)} \quad (2.9)$$

If the initial intensity of the beam of sound is I_o, the mean intensity after traversing a distance y will be

$$I = I_o e^{-\bar{\sigma}(\underline{n}^{(1)})y} \quad (2.10)$$

Formula (2.10) will describe the attenuation correctly only if y is greater than the distance over which there exists appreciable correlation among the eddy sizes (wave-numbers) contributing to the scattering. Within shorter distances, there will be coherence and interference between the waves scattered

from different points. The restriction to distances longer than the correlation length is implicit in (2.9) because the space averaging involved in the correlation tensor $t_{\alpha\beta}(\lambda)$ is customarily carried out, in the case of extended turbulence, over regions large compared with the values of λ of interest.

The Mach number of the turbulence encountered in the atmosphere, the extended medium of principal interest, is sufficiently low that the amount of attenuation in a distance the order of several correlation lengths may be expected to be quite small. In this case, the relative fluctuations to be expected in the attenuation will be very nearly averaged out in a distance of transmission long enough to result in significant weakening of the sound beam. An estimate of the intensity fluctuations may be obtained as follows. Let the linear distance over which the space averaging in (2.6) has been carried out be L. Then the mean scattering cross-section of a region of the turbulence one centimeter square and extending a distance L in the direction of propagation will be $L\bar{\sigma}(\underline{n}''')$. The fluctuations from the mean in the scattering from this region will then be given by $L\Delta\sigma(\underline{n}''')$ where

$$\Delta\sigma(\underline{n}''') = \frac{\pi}{c^2}k_0^4 \int (\underline{n}'''\cdot\underline{n}^{(2)})^2 n_\alpha^{(2)} n_\beta^{(2)} \Delta T_{\alpha\beta}(k,[\underline{n}^{(2)}-\underline{n}''']) d^2\underline{n}^{(2)} \qquad (2.11)$$

The corresponding mean-square fluctuation is

$$\langle[\Delta\sigma(\underline{n}''')]^2\rangle = \frac{\pi^2}{c^4}k_0^8 \iint (\underline{n}'''\cdot\underline{n}^{(2)})^2 (\underline{n}'''\cdot\underline{n}^{(2)\prime})^2 n_\alpha^{(2)} n_\beta^{(2)} n_\mu^{(2)\prime} n_\lambda^{(2)\prime}$$

$$\times \langle\Delta T_{\alpha\beta}(k_0,[\underline{n}^{(2)}-\underline{n}''']) \Delta T_{\mu\lambda}(k_0,[\underline{n}^{(2)\prime}-\underline{n}''']) \rangle d^2\underline{n}^{(2)} d^2\underline{n}^{(2)\prime} \qquad (2.12)$$

where the averaging is over a representative distribution of regions of linear size L.

If the fluctuations in σ are taken into account, the intensity after traversing a distance y will be

$$I = I_0 e^{-\bar{\sigma}(\underline{n}''')y - \int_0^y \Delta\sigma(\underline{n}''')dy} \qquad (2.13)$$

Hence, the mean-square fluctuation in the logarithmic decrement of the intensity will be

$$\langle(\Delta\log\frac{I_0}{I})^2\rangle = \langle(\int_0^y \Delta\sigma(\underline{n}''') dy)^2\rangle \qquad (2.14)$$

It has been assumed that correlation in the turbulence is negligible outside the distance L. Hence if y is considerably larger than L we may expect that the total fluctuation appearing in (2.14) is the result of random addition of fluctuations from y/L independent regions. According to a well-known law we should then have

$$\left\langle \left(\Delta \log \frac{I_0}{I}\right)^2\right\rangle \sim \left(\frac{y}{L}\right)\left\langle \left[\Delta \ \sigma(\underline{n}^{\prime\prime})\right]^2\right\rangle \tag{2.15}$$

so that the ratio of the fluctuation in logarithmic decrement to the mean decrement is

$$\sqrt{\left\langle \left(\Delta \log (I_0/I)\right)^2\right\rangle}/\langle\log (I_0/I)\rangle \sim \left(\frac{L}{y}\right)^{\frac{1}{2}}\sqrt{\left\langle \left(\Delta\sigma(\underline{n}^{\prime\prime})\right)^2\right\rangle}/\bar{\sigma}(\underline{n}^{\prime\prime}) \tag{2.16}$$

It is apparent from (2.12) that the calculation of the fluctuation $\langle[\Delta\sigma(\underline{n}^{\prime\prime})]^2\rangle$ requires a knowledge of the fourth order correlation product

$$S_{\alpha\beta\mu\lambda}(\underline{k},\underline{k}^{\prime}) = \langle \Delta T_{\alpha\beta}(\underline{k})\Delta T_{\mu\beta}(\underline{k}^{\prime})\rangle \tag{2.17}$$

which may be expressed in terms of velocity correlation products by the use of (2.7) and (2.6). The evaluation of such fourth order correlation functions is difficult with existing theoretical and empirical knowledge of turbulence.

2.2   As shown in section 1.1 the phase of a sound wave propagating in a fluid in which there is low Mach number turbulence, the scale of which is large compared to the sound wavelength, becomes distorted according to the eikonal equation

$$\vartheta_{,\alpha}\vartheta_{,\alpha} + 2 c^{-1}\vartheta_{,\alpha}w_{\alpha} - 1 = 0 \tag{1.8}$$

where, if ω is the frequency of the sound wave, $\omega\vartheta/c$ is the total phase. The shear velocity field is w_{α}. If the unperturbed sound is a plane wave propagating in the direction of the unit vector $\underline{n}$, the unperturbed phase will be $\vartheta^{(0)} = \underline{n}\cdot\underline{x}$ so that, writing $\vartheta = \phi + \vartheta^{(0)}$, the eikonal equation may be written

$$2 n_{\alpha}\phi_{,\alpha} + \phi_{,\alpha}\phi_{,\alpha} + 2 c^{-1}(n_{\alpha}w_{\alpha} + \phi_{,\alpha}w_{\alpha}) = 0 \tag{2.18}$$

If the distortion is gradual, corresponding to the assumption of low Mach number, ϕ will vary quite slowly as a function of space and hence the terms $\phi_{,\alpha}\phi_{,\alpha}$ and $\phi_{,\alpha}w_{\alpha}$ will be small compared to

the other terms in (2.18). Therefore, the phase distortion will be determined to a first approximation by the equation

$$n_\alpha \phi_{,\alpha} + c^{-1} n_\alpha w_\alpha = 0 \tag{2.19}$$

If the value of the phase is given on some wavefront by, say, $\phi = 0$, then according to (2.19) the phase further along the direction of propagation will be given by

$$\phi(\underset{\sim}{x}+L\underset{\sim}{n}) = - c^{-1} \int_{\underset{\sim}{x}^0}^{x^0 + nL} w_{(n)}(\underset{\sim}{x}) \, d(\underset{\sim}{x} \cdot \underset{\sim}{n}) \tag{2.20}$$

where $\underset{\sim}{x}^0$ is a point on the initial wavefront, $w_{(n)}(\underset{\sim}{x})$ is the component of $\underset{\sim}{w}$ in the direction of $\underset{\sim}{n}$ and the line integral is over a length L along the direction of $\underset{\sim}{n}$. Although (2.19) is approximately correct everywhere in the region of shear motion, it must be noted that the integrated form (2.20) will be correct only when the change of direction of the wavefronts indicated by $\phi(\underset{\sim}{x})$ is small; i.e., the error introduced in passing from (2.18) to (2.19) is essentially that the initial direction of propagation $\underset{\sim}{n}$, appears in (2.20) instead of the local direction, which is the resultant of the initial direction and the total distortion which has occurred up to the point of interest.

According to (2.20) the mean square phase deviation at the point $\underset{\sim}{x} = \underset{\sim}{x}^0 + \underset{\sim}{n}L$ is

$$\langle [\phi(\underset{\sim}{x})]^2 \rangle = c^{-2} \int_{\underset{\sim}{x}^0}^{\underset{\sim}{x}} \int_{\underset{\sim}{x}^0}^{\underset{\sim}{x}} \langle w_{(n)}(\underset{\sim}{x}') w_{(n)}(\underset{\sim}{x}'') \rangle d(\underset{\sim}{x}' \underset{\sim}{n}) \, d(\underset{\sim}{x}'' \underset{\sim}{n}) \tag{2.21}$$

where the averaging is over a statistically representative ensemble of shear flow systems so as to provide a valid measure of the phase distortion in the turbulent flow. The quantity $\langle w_{(n)}(\underset{\sim}{x}') w_{(n)}(\underset{\sim}{x}'') \rangle$ is just a diagonal component of the two point correlation tensor $\langle w_\alpha(\underset{\sim}{x}') w_\beta(\underset{\sim}{x}'') \rangle$. If the statistics of the turbulence are spatially homogeneous,[6] this tensor is a function of $\underset{\sim}{x}' - \underset{\sim}{x}''$ only and reduces to the single argument correlation tensor defined by (2.6). In this case (2.21) reduces to

$$\langle [\phi(x)]^2 \rangle = c^{-2} \int_{\underset{\sim}{x}^0}^{\underset{\sim}{x}} \int_{\underset{\sim}{x}^0}^{\underset{\sim}{x}} n_\alpha n_\beta t_{\alpha\beta} (\underset{\sim}{x}' - \underset{\sim}{x}'') \, d(\underset{\sim}{x}' \cdot \underset{\sim}{n}) \, d(\underset{\sim}{x}'' \cdot \underset{\sim}{n}) \tag{2.22}$$

(6) This does not imply isotropy.

This expression is easily reduced to single integrals. If we imagine a plane of integration in which $y = \underline{x}' \cdot \underline{n}$ and $z = \underline{x}'' \cdot \underline{n}$ are perpendicular cartesian coordinates, the integration in (2.22) is over a square of side L, oriented with the y and z axes. It is readily seen by drawing the diagonal lines $y - z = \lambda$ that the weight with which the value λ occurs in the square is $L - |\lambda|$. Since λ ranges from L to $-L$ and since $n_\alpha n_\beta t_{\alpha\beta}(\underline{\lambda}) = n_\alpha n_\beta t_{\alpha\beta}(-\underline{\lambda})$ the integral in (2.22) may be rewritten

$$\langle [\phi(\underline{x}^\circ + L\underline{n})]^2 \rangle = 2 c^{-2} n_\alpha n_\beta \int_o^L t_{\alpha\beta}(\lambda\underline{n})(L-\lambda)\,d\lambda \qquad (2.23)$$

where we have noted in writing $t_{\alpha\beta}(\lambda\underline{n})$ that the integration is along $\underline{n}$.

When the distance of propagation L is considerably longer than the maximum distance over which there exists correlation in the turbulence, $t_{\alpha\beta}(\lambda\underline{n})$ will be negligible for values of λ greater than L and hence, for this case,[7]

$$\langle [\phi(\underline{x}^\circ + L\underline{n})]^2 \rangle \sim 2 c^{-2} n_\alpha n_\beta \left(L \int_o^\infty t_{\alpha\beta}(\lambda\underline{n})\,d\lambda - \int_o^\infty t_{\alpha\beta}(\lambda\underline{n})\lambda\,d\lambda \right) \qquad (2.24)$$

Under the assumption that L is large, the second term will be small relative to the first and hence the plausible result that the root-mean-square phase deviation varies as $\sqrt{L}$, and as a quantity associated with the mean velocity fluctuation length and magnitude.

According to section 1.1 the local intensity of the distorted wave (more accurately, the square of the amplitude) is related to the phase deviation by the continuity equation

$$\epsilon_{,\alpha}\vartheta_{,\alpha} = -\nabla^2\vartheta - 2c^{-1}\vartheta_{,\alpha}\vartheta_{,\beta}W_{\alpha,\beta} \qquad (1.9)$$

where ϵ is the logarithm of the intensity. Under the approximation employed in deriving the phase deviation we may replace (1.9) by the approximate form

$$\epsilon_{,\alpha}n_\alpha = -\nabla^2\phi - 2c^{-1}n_\alpha n_\beta W_{\alpha,\beta} \qquad (2.25)$$

(7) A similar derivation was carried out for a specific model of turbulent flow by Blockintzev: "Acoustics of an Inhomogeneous Moving Medium." Translation by R. T. Beyer and D. Mintzer under RAG Contract N7-ONR-35808 (August 1952)

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so that, following the analysis just given,

$$\epsilon(\underline{x}^0 + L\underline{n}) = -\int_{\underline{x}^0}^{\underline{x}^0 + L\underline{n}} [\nabla^2 \phi(\underline{x}) + 2c^{-1} n_\alpha n_\beta w_{\alpha,\beta}] \, d(\underline{x} \cdot \underline{n}) \qquad (2.26)$$

will represent the approximate logarithmic intensity increment after propagating a distance L into the moving medium. Writing out ∇^2 in the form $\partial^2/(\partial x_1)^2 + \partial^2/(\partial x_2)^2 + \partial^2/(\partial x_3)^2$ where $\underline{n}$ lies along the x_3 axis, and using expression (2.20) for $\phi(\underline{x})$,

$$\nabla^2 \phi(\underline{x}) = - c^{-1} \int_{\underline{x}^0}^{\underline{x}} \left[\frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2} \right] w_3(\underline{x}) \, dx_3 - c^{-1} \frac{\partial w_3(\underline{x})}{\partial x_3} \qquad (2.27)$$

Hence,

$$\epsilon(\underline{x}^0 + L\underline{n}) = -\int_{\underline{x}^0}^{\underline{x}^0 + L\underline{n}} \left[\frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2} \right] \phi(\underline{x}) \, dx_3 - \frac{w_3(\underline{x}^0 + L\underline{n}) - w_3(\underline{x}^0)}{c} \qquad (2.28)$$

The term $(w_3(\underline{x}^0 + L\underline{n}) - w_3(\underline{x}^0))/c$ depends only upon variations, in the direction of propagation, of w_3, the component of shear velocity in the direction of propagation. Its physical significance is that it represents the change in amplitude required to maintain a constant transfer of energy as the velocity of propagation along a line of propagation varies according to $w_3(\underline{x})$ It is apparent that this term is of a fluctuating nature, and in view of the spatial homogeneity of the turbulence which we have assumed, it will not make any cumulative contribution to $\epsilon(\underline{x}^0 + L\underline{n})$, and will not be considered further.

The first term, involving derivatives with respect to x_1 and x_2, represents changes in local intensity due to fluctuations in the spacing of ray paths as a result of the curvature of the phase fronts. In fact, $\partial^2\phi/(\partial x_1)^2 + \partial^2\phi/(\partial x_2)^2$ gives just the local curvature of a phase front. This type of intensity fluctuation may be regarded as the resultant of a series of localized focusing and defocusing actions. Let the integral in (2.28) be denoted by $\epsilon_T(\underline{x})$. Then, if we employ the notation

$$[\partial^2/(\partial x_1)^2 + \partial^2/(\partial x_2)^2] f(\underline{x}) \equiv f^{II}(\underline{x}) \qquad (2.29)$$

for compactness,

$$\langle [\epsilon_T(\underline{x}^0 + L\underline{n})]^2 \rangle = \int_{\underline{x}^0}^{\underline{x}^0 + L\underline{n}} \int_{\underline{x}'}^{\underline{x}^0 + L\underline{n}} \langle \phi^{II}(\underline{x}) \phi^{II}(\underline{x}') \rangle \, dx_3 \, dx_3' \qquad (2.30)$$

$$(2.30)$$

represents the mean square intensity fluctuation due to wave front curvature. Using (2.20),

$$\langle \phi^{II}(\underline{x}) \phi^{II}(\underline{x}') \rangle = c^{-2} \int_{\underline{x}^\circ}^{\underline{x}} \int_{\underline{x}^\circ}^{\underline{x}'} \langle w_3^{II}(\underline{x}) w_3^{II}(\underline{x}') \rangle \, dx_3 \, dx_3' \qquad (2.31)$$

Writing

$$\langle w_3^{II}(\underline{x}) w_3^{II}(\underline{x}') \rangle = c^2 f(\underline{x} - \underline{x}') \qquad (2.32)$$

we note that this correlation product can depend only on the difference of the two coordinate points in the case of homogeneous turbulence; then,

$$\langle \phi^{II}(\underline{x}) \phi^{II}(\underline{x}') \rangle = \int_0^{y} \int_0^{y'} f(\underline{x} - \underline{x}') \, dx \, dx' \qquad (2.33)$$

where $y = \underline{x} - \underline{x}^\circ$, $y' = \underline{x}' - \underline{x}^\circ$, and the variables of integration have been transformed to give more convenient limits. The integral in (2.33) is similar to that in (2.22) except that the area of integration is a rectangle in this case instead of a square. Using a geometrical construction similar in principle to that employed in the evaluation of (2.22), one finds, with some manipulation,

$$\langle \phi^{II}(\underline{x}^\circ + \underline{y}) \phi^{II}(\underline{x}^\circ + \underline{y}') \rangle = \int_0^{y} f(\lambda \underline{n})(y - \lambda) d\lambda + \int_0^{y'} f(\lambda \underline{n})(y' - \lambda) d\lambda \\ - \int_0^{y-y'} f(\lambda \underline{n})(y - y' - \lambda) d\lambda \qquad (2.34)$$

It will be noted that this expression is symmetric in y and y'. Denoting it by $G(y, y')$,

$$\langle [\epsilon_T (\underline{x}^\circ + L\underline{n})]^2 \rangle = \int_0^L \int_0^L G(y, y') \, dy \, dy' \qquad (2.35)$$

As in the case of the phase deviation, this expression for the intensity fluctuations in terms of integrals over second order velocity derivative correlation products can be simplified when the propagation distance L is much longer than distances over which there exists correlation in the turbulence. For such long distances, following the argument used in obtaining the approximation (2.24),

$$G(y, y') \sim (y + y') \int_0^\infty f(\lambda \underline{n}) d\lambda - |y - y'| \int_0^\infty f(\lambda \underline{n}) d\lambda \qquad (2.36)$$

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Approximately correct results will be obtained by substituting this asymptotic form for $G(y, y')$ into (2.35). The reason is that since $G(y, y')$ grows with y and y', the contribution to (2.35) from values so small that (2.36) is not valid will be minor. Hence setting

$$F = \int_0^\infty f(\lambda \underline{n}) d\lambda \qquad (2.37)$$

we have

$$\langle [\epsilon_T (\underline{x}^\circ + L\underline{n})]^2 \rangle \sim F \int_0^L \int_0^L [(y+y') - |y-y'|] \, dy \, dy' \qquad (2.38)$$

Considering this as an integral over a square of side L, it may be handled as were the previous cases, except that for evaluation of the term in $y+y'$, the weighting of the lines $y+y'=\lambda$ will be $L-|L-\lambda|$, and λ will range from 0 to $2L$. Then,

$$\int_0^L \int_0^L [(y+y')-|y-y'|] dy \, dy' = \int_0^{2L} \lambda (L-|L-\lambda|) \, d\lambda - 2 \int_0^L \lambda (L-\lambda) d\lambda = \frac{2}{3} L^3 \qquad (2.39)$$

Hence,

$$\langle [\epsilon_T (\underline{x}^\circ + L\underline{n})]^2 \rangle \sim \frac{2}{3} L^3 F \qquad (2.40)$$

where F is defined in terms of the quadratic correlation of the velocity derivatives by (2.37), (2.32), (2.29). Thus the root-mean-square intensity fluctuation will increase as $L^{3/2}$. The rapid increase of the intensity fluctuations relative to the phase deviation is due to the fact that the fluctuation at distance is the cumulative result of wavefront curvature along the entire path of propagation as well as at the point L.

Part 3. <u>Scattering by Anisotropic Turbulence</u>

In a previous paper[1] an explicit expression was derived for the scattering of sound in a region containing isotropic turbulence of arbitrary spectrum shape. In this section, the effects of anistropy on the scattering are discussed.

3.1   In most physical examples of turbulence, there is a significant degree of anisotropy because the mechanism by which energy is supplied to the turbulence tends to produce vorticity in certain preferred directions. In the process of transfer of the energy from large eddies to small the anistropy becomes obscured gradually, and near the upper limit of the turbulence spectrum there is usually a high degree of isotropy. In terms of spatial distribution, it may be expected that the maximum anistropy exist near the vorticity producing boundaries while

as the vorticity diffuses away from the boundary, the anisotropy become obscured. This is true, for example, in the case of atmospheric turbulence resulting from vorticity produced at the ground.

In speaking of the isotropy or anisotropy of the turbulence in a region of fluid we refer not to the instantaneous character of the flow but to statistical properties which are defined as averages over either a long period of time, or, as is practically equivalent in this case, a very large volume of fluid containing many such regions as we are examining. Alternatively, the statistical characteristics may be expressed as averages over an ensemble of flow system, the ensemble being so chosen as to represent the fluctuations encountered in time averaging or large scale space averaging. In what follows the ensemble average of a quantity A will be denoted by $\langle A \rangle$.

Consider a shear motion for which the velocity is given by $W_\alpha(\underline{x}, t)$. The velocity field may be described in terms of a Fourier representation by

$$W_\alpha(\underline{k}, \omega) = (2\pi)^{-2} \int W_\alpha(\underline{x}, t) e^{-i(\omega t - k_\alpha x^\alpha)} \, d^3\underline{x} \, dt \qquad (3.1)$$

The condition that the motion be a pure shear flow is

$$\frac{\partial W_\alpha(\underline{x})}{\partial x^\alpha} = 0 \qquad (3.2)$$

or

$$k_\alpha W_\alpha(\underline{k}, \omega) = 0 \qquad (3.3)$$

In the reference mentioned,[1] it was shown that when a plane sound wave of frequency ω_0 propagating in the direction of the unit vector $\underline{n}^{(1)}$ is scattered in a region of a fluid undergoing such shear flow at low Mach number, the power spectrum of the energy scattered per unit solid angle in the direction $\underline{n}^{(2)}$ is given by

$$I(\underline{n}^{(1)}, \underline{n}^{(2)}, \omega) = \frac{\pi \omega^4}{c^6} (\underline{n}^{(1)} \cdot \underline{n}^{(2)})^2 \left| n_\sigma^{(2)} W_\sigma \left(\frac{|\omega| \underline{n}^{(2)} - \omega_0 \underline{n}^{(1)}}{c}, |\omega| - \omega_0 \right) \right|^2 \qquad (3.4)$$

$I(\underline{n}^{(1)}, \underline{n}^{(2)}, \omega)$ is so normalized that the differential scattering cross-section is given by

$$\sigma(\underline{n}^{(1)}, \underline{n}^{(2)}) = \frac{1}{T} \int_{-\infty}^{\infty} I(\underline{n}^{(1)}, \underline{n}^{(2)}, \omega) \, d\omega \qquad (3.5)$$

where τ is the interval over which the time integration in (3.1) is carried out.

In order to investigate the effects of anisotropy on the scattering predicted by (3.4), it is convenient to express the Fourier elements $W_\sigma(k,\omega)$ in terms of the vorticity. The vorticity axial vector is defined by

$$\Psi_\sigma(x,t) = \epsilon_{\sigma\alpha\beta}\,\partial W_\alpha(x,t)/\partial x^\beta$$

$$\Psi_\sigma(k,\omega) = -i\,\epsilon_{\sigma\alpha\beta}\,k_\beta W_\alpha(k,\omega)$$

(3.6)

where the quantity $\epsilon_{\sigma\alpha\beta}$ is antisymmetric to permutation of any two indices and $\epsilon_{123}=1$. From this it follows,[8] as a result of (3.3), that

$$W_\sigma(k,\omega) = i\,\epsilon_{\sigma\alpha\beta}\,k^{-2}k_\alpha\,\Psi_\beta(k,\omega)$$

(3.7)

Substituting this expression in (3.4),

$$I(n^{(1)},n^{(2)},\omega) = \frac{\pi\omega^4}{c^6}(n^{(1)}\cdot n^{(2)})^2 \left|k^{-2}\epsilon_{\sigma\alpha\beta}\,n^{(2)}_\sigma\,(|\omega|n^{(2)}_\alpha - \omega_o n^{(1)}_\alpha)\,\Psi_\beta\left(\frac{|\omega|n^{(2)}-n^{(1)}\omega_o}{c},|\omega|-\omega_o\right)\right|^2$$

with $k = ||\omega|n^2 - \omega_o n^{(1)}|$.

Because of the antisymmetry of $\epsilon_{\sigma\alpha\beta}$ in α and σ the term in $|\omega|n^{(2)}_\alpha$ vanishes, leaving

$$I(n^{(1)},n^{(2)},\omega) = \frac{\pi\omega^4\omega_o^2}{c^6}(n^{(1)}\cdot n^{(2)})^2 \left|\epsilon_{\sigma\alpha\beta}\,n^{(2)}_\sigma\,n^{(1)}_\alpha\,\Psi_\beta\left(\frac{|\omega|n^{(2)}-\omega_o n^{(1)}}{c},|\omega|-\omega\right)\right|^2$$

$$\times \left(|\omega|n^2 - \omega_o n^{(1)}\right)^{-4}$$

(3.8)

or, introducing the vector velocity potential $V_\sigma(x,t)$ by

$$W_\sigma(k,\omega) = i\,\epsilon_{\sigma\alpha\beta}\,k_\beta V_\alpha(k,\omega)$$

$$W_\sigma(x,t) = \epsilon_{\sigma\alpha\beta}\,\partial V_\alpha(x,t)/\partial x^\beta$$

(3.9)

so that

$$\Psi_\sigma(x,t) = \nabla^2 V_\sigma(x,t) \qquad \Psi_\sigma(k,t) = -k^2 V_\sigma(k,t)$$

(3.10)

(8) This is readily shown by multiplying (3.6) by $k_\lambda\epsilon_{\omega\lambda\sigma}$ and using the relation $\epsilon_{\sigma\mu\lambda}\epsilon_{\sigma\alpha\beta} = \delta^\mu_\alpha\delta^\lambda_\beta - \delta^\lambda_\alpha\delta^\mu_\beta$.

(3.8) becomes

$$I(\underline{n}^{(1)}, \underline{n}^{(2)}, \omega) = \frac{\pi \omega^4 \omega_o^2}{c^6} |\epsilon_{\sigma \kappa \rho} \, n_{\tau}^{(2)} n_{\alpha}^{(1)} V_{\rho} \left(\frac{|\omega| \underline{n}^{(2)} - \omega_o \underline{n}^{(1)}}{c}, |\omega| - \omega_o \right)|^2 \tag{3.11}$$

This form, or (3.8) displays the angular behavior of the scattering more immediately than the original expression (3.4), and shows that $\underline{n}^{(2)}$ and $n^{(1)}$ play symmetrical roles in the scattering formula. The quantity in the straight brackets has a very simple geometrical significance. It is equal to the volume of the parallelopided formed by the vectors $\underline{n}^{(2)}$, $\underline{n}^{(1)}$ and $\underline{V}$. Thus the angular dependence, aside from the detailed structure of $\underline{V}$, is the combined result of two factors; first, the quantity $\underline{n}^{(1)} \cdot \underline{n}^{(2)}$ which favors scattering in forward and backward directions while indicating zero $90°$ scattering, and second, the parallelopided volume which is maximum when the angles between the three vectors are right angles and hence indicates zero scattering at zero[9] and $180°$, and also in the case when either $\underline{n}^{(1)}$ or $\underline{n}^{(2)}$ lies in the direction of the vorticity component for the particular wave number involved. Thus in the case of a shear flow in which there is a preferred direction of vorticity it is to be expected that the total scattering will be maximum when the direction of the incident wave is normal to the vorticity direction and that the scattered radiation will be most intense in or near the plane normal to the vorticity. This specific angular behavior would be superimposed on the angular dependence associated with isotropic turbulence.

Both the vorticity $\underline{Y}$ and the velocity potential $\underline{V}$ have vanishing divergence if the velocity field itself is to be divergenceless. However, the divergenceless character of $\underline{w}$ as defined in (3.9) is independent of any condition on $\underline{V}$. Hence, (3.11) will describe the scattering from some possible incompressible flow, irrespective of whether or not $\underline{V}$ is divergenceless. In the latter case, however, the vorticity is not related to $\underline{V}$ in the simple fashion indicated in (3.10).

The advantage of describing the shear velocity field by the inherently divergenceless expression (3.9) is that one need not impose explicitly on the statistical ensembles describing turbulent flows the condition (3.3) that the velocity vector and wave vector be perpendicular. As an example, for which no observational validity is claimed, we shall examine the scattering from a velocity distribution which describes a particular type of hypothetical turbulence with a preferred axis of vorticity and which leads to especially simple expressions. Consider the

(9) Strictly speaking, it cannot be immediately concluded that the scattering at zero degrees vanishes without ascertaining the behavior of $\underline{V}(\underline{k}, \omega)$ for very small $\underline{k}$. It is apparent, however, as was pointed out in reference (1), that the frequency spectrum of the radiation in the forward direction must be very sharp.

velocity spectrum

$$W_\sigma(\underline{k}, \omega) = \epsilon_{\sigma\alpha\beta} k_\alpha n_\beta^{(0)}(\underline{k}, \omega) f(\underline{k}, \omega) \qquad (3.12)$$

in which $\underline{n}^{(0)}$ is a unit vector. For a fixed, constant $\underline{n}^{(0)}$, this spectrum describes motion confined to directions lying in the plane perpendicular to $\underline{n}^{(0)}$. If it is assumed that the ensemble distribution of $\underline{n}^{(0)}$ is independent of $\underline{k}, \omega$ and that the anisotropy of the turbulence is completely described by the distribution of $\underline{n}^{(0)}$; i.e., if $\langle |f(k,\omega)|^2 \rangle$ depends only on k^2, ω and $f(\underline{k}, \omega)$ is uncorrelated with $\underline{n}^{(0)}$, the ensemble average of the scattering will be given by

$$\langle J(\underline{n}^{(1)}, \underline{n}^{(2)}, \omega) \rangle = \frac{\pi \omega^4 \omega_0^2}{c^6} (\underline{n}^{(1)} \cdot \underline{n}^{(2)})^2 \langle |\epsilon_{\sigma\alpha\beta} n_\sigma^{(2)} n_\alpha^{(1)} n_\beta^{(0)}|^2 \rangle F(|\frac{\omega \underline{n}^{(2)} - \omega_0 \underline{n}^{(1)}}{c}|, |\omega| - \omega_0) \quad (3.13)$$

where $\quad F(|k|, \omega) = \langle |f(k, \omega)|^2 \rangle$

For this situation it is apparent that the dependence of the scattering on the direction of the incident wave, $\underline{n}^{(1)}$, is entirely contained in the factor $\langle |\epsilon_{\sigma\alpha\beta} n_\sigma^{(2)} n_\alpha^{(1)} n_\beta^{(0)}| \rangle$ which is the mean square volume of the parallelopiped bounded by the three unit vectors. This quantity is easily calculated for a given distribution of $\underline{n}^{(0)}$ around some preferred direction. In view of the simplicity of this example it would be of interest to see to what extent axially symmetric turbulent flow occurring in practice could be fitted to the distribution described. The principal lack of generality in the axially symmetric distributions described by (3.12) with the specializations we have noted is that a definite weighting is given to wave-vectors $\underline{k}$ according to the angle they make with the axial direction $\underline{n}^{(0)}$, as may be evidenced by forming the energy-spectrum corresponding to (3.12):

$$|\underline{W}(\underline{k}, \omega)|^2 = [k^2 - c(\underline{k} \cdot \underline{n}^{(0)})^2] |f(k, \omega)|^2 \qquad (3.14)$$

This weighting, which favors wave-vectors lying in the preferred plane of motion, is not an unreasonable one, a priori, but it should be emphasized that no empirical validation of this distribution is implied.

Appendix

In the analysis of Section 2.2 there appeared several integrals of the form

$$I_1 = \int_0^a \int_0^b G(|x-y|) dx \, dy \qquad (A.1)$$

or

$$I_2 = \int_0^a \int_0^a H(x+y) dx \, dy \qquad (A.2)$$

These expressions are readily converted to single integrals by first finding the weight with which given values of the arguments $\lambda = |x-y|$ or $\lambda = x+y$ occur in the rectangular regions of integration. Considering I_1, first, it appears from Figure 1 that in region II of the domain of integration the weight corresponding to the argument value λ is $b \, d\lambda$, the area of the parallelogram of base $d\lambda$ and height b. Similarly, neglecting areas of order $(d\lambda)^2$, the weights appropriate to regions I and III are $(b-|\lambda|)d\lambda$ and $(a-\lambda)d\lambda$ respectively. In region I, $|\lambda|$ varies from o to b, in region II from o to $a-b$, and in region III from $a-b$ to a. Hence I_1, expressed as the sum of integrals over the three regions, is given by

$$I_1 = \int_0^b G(\lambda)(b-\lambda)d\lambda + \int_0^{a-b} G(\lambda)b \, d\lambda + \int_{a-b}^a G(\lambda)(a-\lambda)d\lambda \qquad (A.3)$$

If we note that

$$\int_0^{a-b} G(\lambda)b \, d\lambda = \int_0^{a-b} G(\lambda)(a-\lambda)d\lambda - \int_0^{a-b} G(\lambda)(a-b-\lambda)d\lambda \qquad (A.4)$$

this may be written in the form,

$$I_1 = \int_0^b G(\lambda)(b-\lambda)d\lambda + \int_0^a G(\lambda)(a-\lambda)d\lambda - \int_0^{a-b} G(\lambda)(a-b-\lambda)d\lambda \qquad (A.5)$$

This expression is symmetric in a and b (as is readily verified for the last term) and hence is valid also for $a < b$. When $a = b$,

$$I_1 = 2 \int_0^a G(\lambda)(a-\lambda)d\lambda \qquad (A.6)$$

which agrees with the result of the derivation described in the text for the integration over a square.

Considering now the evaluation of I_2, it appears from Figure 2 that in region I the weight corresponding to the argument value $x+y=\lambda$ is $\lambda d\lambda$ while in region II it is $(2a-\lambda)d\lambda$. In region I λ ranges from 0 to a and in region II, from a to $2a$. Hence, both these weights can be expressed by the single function $(a-|a-\lambda|)d\lambda$. Then,

$$I_2 = \int_0^{2a} H(\lambda)(a-|a-\lambda|)d\lambda \qquad (A.7)$$

This method of evaluating I_2 can be extended to rectangular domains of integration also.

References

(1) Kraichnan, Robert H., J.A.S.A. <u>25</u> 1096 (1953)

(2) Lighthill, M. J., Proc. Royal Soc. <u>A211</u> 564 (1952),
         Proc. Camb. Phil. Soc. <u>49</u> 531 (1953)

(3) Goldstein, Herbert, "Classical Mechanics" Cambridge:
         Addison-Wesley (1951)

(4) Blockintzev, "Acoustics of an Inhomogeneous Moving
         Medium." Translation by R. T. Bayer and D. Mintzer
         under RAG Contract N7-ONR-35808, Brown University,
         Providence (August 1952)

FIGURE 1. DOMAIN OF INTEGRATION FOR I_1

FIGURE 2. DOMAIN OF INTEGRATION FOR I_2